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A MOMENT METHOD FOR COMPRESSIBLE LAMINAR BOUNDARY LAYERS AND SOME APPLICATIONS†

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Abstract—An integral method based on moment techniques and especially useful for compressible laminar flows is presented. The describing partial differential equations are first transformed to the Levy–Lees variables and then converted to integral conditions with η^m as a weighing factor. A technique for handling the variation in transport properties which explicitly arises for $m > 0$ is described. The resultant equations are applied for $m = 0, 1$ with the commonly employed fourth and fifth degree polynomial profiles for velocity and stagnation enthalpy. The distinct classification of similar and nonsimilar flows within the same analytic framework is emphasized. The analysis is applied to a variety of flows for which more accurate results are available and is found to yield satisfactory results in most cases. Therefore, the present method is considered to improve somewhat the conventional integral method ($m = 0$) without excessive labor.

NOMENCLATURE

a_2, velocity profile parameter;
A_{ij}, square matrix elements [see equation (34)];
b_1, b_2, energy profile parameters;
B_i, column matrix elements [see equation (34)];
c_f, skin friction coefficient;
C, mass–density viscosity ratio, $\rho\mu/\rho_e\mu_e$;
$C_{0,m}$, mean value of C [see equations (4)–(6)];
$\tilde{C}$, $= C/\sigma$;
$\tilde{C}_{0,m}$, mean value of $\tilde{C}$ [see equations (16)–(18)];
$\bar{C}$, $= C(1 - \sigma^{-1})$;
$\bar{C}_{0,m}$, = mean value of $\bar{C}$ [see equations (20)–(22)];
f, transformed stream function;
f_η, streamwise velocity ratio, (u/u_e);
g, stagnation enthalpy ratio, $h_s/h_{s,e}$;
G, shear parameter;
h, enthalpy;
I_i, various integrals of profiles, [see Appendix];
j, $= 0, 1$ for two-dimensional or axisymmetric flow respectively;
m, exponent of weight function;
$\tilde{m}$, $= (u_e^2/2h_{s_e})$;
M, Mach number;
p, pressure;
q, heat transfer;
r, cylindrical radius;
$\tilde{s}$, transformed streamwise variable;
$\bar{s}$, normalized streamwise variable for blunt body problem;
u, streamwise velocity;
x, coordinate in streamwise direction;
y, coordinate normal to streamwise direction;
β, pressure gradient parameter;
$\bar{\beta}$, constant of proportionality, $\bar{u} = \bar{\beta}\bar{x}$, for blunt body problem;
δ, boundary-layer thickness;
δ^*, boundary-layer displacement thickness;
η, transformed normal coordinate;
$\tilde{\eta}$, $= \eta/\eta_e$;

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θ, boundary-layer momentum thickness;
μ, viscosity;
ρ, mass density;
σ, Prandtl number, $(\mu C_p/k)$;
τ, skin friction.

Subscripts
e, external condition;
i, initial conditions;
0, reference conditions;
s, e, external stagnation conditions;
$\tilde{s}$, differentiation with respect to $\tilde{s}$, $(\partial/\partial\tilde{s})$;
w, wall conditions;
η, differentiation with respect to η, $(\partial/\partial\eta)$;
∞, conditions far from the body.

Superscript
$()'$ differentiation with respect to $\tilde{s}$, $(\partial/\partial\tilde{s})$.

I. INTRODUCTION

PROBLEMS in the theory of laminar boundary layers continue to occupy the attention of fluid mechanicians in a wide variety of fields. Consequently, there exists a broad literature providing various methods of solution of such problems. The greatest advances in the theory in recent years have been connected with so-called similar flows which are described by ordinary differential equations involving as a new independent variable a combination of dependent and independent variables (cf. e.g. [1–5]). However, many applied problems involve either combinations of wall and external flow conditions and of initial distributions of flow variables incompatible with similarity restrictions or homogeneous chemical reactions with finite rates. Thus there is continuing interest in nonsimilar flows, which are amenable to treatment by a spectrum of methods from exact numerical procedures to conventional integral methods. With respect to the former procedures, references 6–8 are representative of the present status thereof.

The integral methods, although widely used, have been criticized on at least two counts when applied to flows of interest in aeronautics and astronautics. It is objected that as a result of the averaging over the boundary-layer region, the only transport properties remaining are those at the wall; in flows involving large density gradients correspondingly large changes in viscosity, conductivity and diffusivity occur in the boundary layer and are not accounted for in the usual integral method. The second objection pertains to the difficulty in systematically improving solutions so that there can be established some measure of the accuracy of the approximation being employed. Although such improvements are in principle possible in many integral methods, the complexity of doing so usually precludes such evaluations. In this connection the strip method of Pallone [9, 10] must be mentioned since it might be considered intermediate between the exact numerical procedures, which it approaches when the number of strips become large, and the conventional integral method, which it reduces to when only one strip is employed; accordingly, it largely overcomes the aforementioned objections to the integral method.

It is the purpose of this report to present an integral method based on the method of moments; it also meets the objections cited above and may thus be considered an alternative to the strip method although perhaps more closely akin to the conventional integral method; i.e. in the spectrum of approximate methods it should be considered near the more approximate, more simple end.

Moment methods have been employed in the past in connection with approximate solutions to boundary-layer problems (cf. e.g. [11–16]) but principally to constant density flows. With respect to compressible flows the transformation of the equations according to the Howarth–Dorodnitzn technique requires that the velocity ratio (u/u_e) be used as a weighing factor in the moment method. This complicates the evaluation of the integrals arising in the various moment conditions and increases the complexity of taking higher moments.† Following [5] and [17], the boundary-layer equations are here first transformed to the Levy–Lees variables $\tilde{s}$, η. Moment conditions are then imposed with η^m taken as a weighing factor; accordingly, the

† The first author is indebted to Mr. Clark H. Lewis of ARO, Inc., Tullahoma, Tennessee, for an informative discussion of the difficulties of moment methods in compressible flows.

possibility of adding additional moment conditions is enhanced. An additional advantage of this approach is that it leads to a clear characterization of similar and nonsimilar flows.

This report is organized as follows: the general analysis leading to sets of moment conditions is presented first; a particular pair of profiles, one for the velocity and one for the stagnation enthalpy is then assumed so that the implications of the resulting set of equations can be discussed. Finally, several applications of the analysis employing these profiles are compared with the results of more accurate calculations.

The particular profiles employed here are of the well-known polynomial form although the general analysis can be applied to other types of profile representations. As in most integral methods involving polynomial profiles, there arise in the present work limitations of the method related to essential singularities in the final system of equations; thus, for flows with accelerating pressure gradients there is a maximum value of g_w beyond which the solutions cannot be carried to arbitrary downstream distances and for flows with suction the extent and magnitude of the mass transfer is restricted. Accordingly, the main interest of the present work is considered to be the general analysis including the treatment of transport property effects and of similar and nonsimilar flows within the same analytic framework.

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II. ANALYSIS

Consider first the streamwise momentum equation in terms of the modified stream function $f(\tilde{s}, \eta)$ with the Levy–Lees variable $\tilde{s}, \eta$ [5]; it is

$$(Cf_{\eta\eta})_\eta + ff_{\eta\eta} + \beta\,[(\rho_e/\rho) - f_\eta^2] = 2\,\tilde{s}\,(f_\eta f_{\eta\tilde{s}} - f_{\tilde{s}} f_{\eta\eta}) \qquad (1)$$

For the present analysis the boundary conditions applicable for all $\tilde{s}$ are taken to be

$$\left.\begin{aligned} f(\tilde{s}, 0) &= f_w(\tilde{s}) \\ f_\eta(\tilde{s}, 0) &= 0 \\ f_\eta(\tilde{s}, \infty) &= 1 \end{aligned}\right\} \qquad (2)$$

No detailed specification of initial conditions applicable either at $\tilde{s} = 0$ or at $\tilde{s} = \tilde{s}_i > 0$ will be listed at this juncture; however, the usual restrictions relative to solutions valid as $\tilde{s} \to 0$ will be considered acceptable. The transport parameter C is considered here a given function of the dependent variables, e.g. of f_η and g and β is considered a given function of $\tilde{s}$.

Integral conditions

To obtain an approximate solution to equations (1) and (2) introduce a finite "edge" $\eta = \eta_e = \eta_e(\tilde{s})$ at which $f_\eta = f_\eta(\tilde{s}, \eta_e) \simeq 1$ and obtain a series of integral conditions by multiplying equation (1) by $\eta^m\,d\eta$ and by integrating termwise from 0 to η_e. There will result from this procedure a system of ordinary differential equations which must be integrated numerically. It should be noted that in formulating the integral conditions it will be assumed that a velocity profile in the form $f_\eta = \sum_{i=1}^{N} a_i(\tilde{s})F_i(\tilde{\eta})$ where $\tilde{\eta} \equiv \eta/\eta_e$ is anticipated so that derivatives of the type $(f_\eta)_{\tilde{\eta}}$ will appear and will be conveniently retained in this form. The $F_i(\tilde{\eta})$ functions will be compatible at least with the boundary conditions of equation (2).

The first term on the left-hand side of equation (1) becomes

$$\begin{aligned} \int_0^{\eta_e} \eta^m\,(Cf_{\eta\eta})_\eta\,d\eta &= -\,[C\,(f_\eta)_\eta]_w = \\ &-\,C_w\,[(f_\eta)_{\tilde{\eta}}]_w/\eta_e, \quad m = 0 \\ &= -\,m\,\eta_e^{m-1}\int_0^1 \tilde{\eta}^{m-1}\,C\,(f_\eta)_{\tilde{\eta}}\,d\tilde{\eta}, \quad m \geqslant 1 \qquad (3) \end{aligned}$$

It will be noted that for $m \geqslant 1$ the influence of the transport properties throughout the boundary layer is present. In anticipation of the numerical analysis related to the integrals involving C, it will be convenient to introduce a mean value of C, denoted $C_{0,\,m}$ and varying with $\tilde{s}$ and to consider the remainder to be a function of $\tilde{s}$ and η, known from a previous step in the integration of the ordinary differential equations.

In particular let

$$C(\tilde{s}, \eta) = [C(\tilde{s}, \eta) - C_{0,m}(\tilde{s})] + C_{0,m}(\tilde{s}) \quad (4)$$

Then equation (3) for $m \geq 1$ becomes

$$\int_0^{\eta_e} \eta^m (C f_{\eta\eta})_\eta \, d\eta = - m \eta_e^{m-1} C_{0,m} [1 - (m - 1) \int_0^1 \tilde{\eta}^{m-2} f_\eta \, d\tilde{\eta} + \int_0^1 \tilde{\eta}^{m-1} (C - C_{0,m}) C_{0,m}^{-1} (f_\eta)_{\tilde{\eta}} \, d\tilde{\eta}] \quad (5)$$

Now select $C_{0,m}$ so that the second integral is zero, i.e. let

$$C_{0,m} = \frac{\int_0^1 \tilde{\eta}^{m-1} C (f_\eta)_{\tilde{\eta}} \, d\tilde{\eta}}{1 - (m-1) \int_0^1 \tilde{\eta}^{m-2} f_\eta \, d\tilde{\eta}} \quad (6)$$

so that finally

$$\int_0^{\eta_e} \eta^m (C f_{\eta\eta})_\eta \, d\eta = - m \eta_e^{m-1} C_{0,m} [1 - (m - 1) \int_0^1 \tilde{\eta}^{m-2} f_\eta \, d\tilde{\eta}], \quad m \geqslant 1 \quad (7)$$

The remaining terms on the left-hand side of equation (1) offer no difficulty; they become

$$\int_0^{\eta_e} \eta^m f f_{\eta\eta} \, d\eta = \eta_e^m [f_e - \eta_e \int_0^1 \tilde{\eta}^m f_\eta^2 \, d\tilde{\eta} - m \int_0^1 \tilde{\eta}^{m-1} f f_\eta \, d\tilde{\eta}] \quad (8)$$

The function $f(\tilde{s}, \eta)$ will be computed from the profile f_η from the equation

$$f = f_w + \eta_e \int_0^{\tilde{\eta}} f_\eta \, d\tilde{\eta} \quad (9)$$

and the function $f_e(\tilde{s})$ from

$$f_e = f_w + \eta_e \int_0^1 f_\eta \, d\tilde{\eta} \quad (10)$$

$$\beta \int_0^{\eta_e} \eta^m [(\rho_e/\rho) - f_\eta^2] \, d\eta = \beta \eta_e^{m+1} [\int_0^1 \tilde{\eta}^m (\rho_e/\rho) \, d\tilde{\eta} - \int_0^1 \tilde{\eta}^m f_\eta^2 \, d\tilde{\eta}] \quad (11)$$

In cases in which ρ_e/ρ is simply related to the stagnation enthalpy and velocity the integral involving the density can be handled analytically; in more complicated cases this integral requires numerical treatment.

The right-hand side of equation (1) becomes

$$2\tilde{s} \int_0^{\eta_e} \eta^m (f_\eta f_{\eta\tilde{s}} - f_{\tilde{s}} f_{\eta\eta}) \, d\eta$$

$$= 2\tilde{s}\, \eta_e^m \left\{ \left(\frac{d\eta_e}{d\tilde{s}} \right) \left[(m+1) \int_0^1 \tilde{\eta}^m f_\eta^2 \, d\tilde{\eta} - \int_0^1 f_\eta \, d\tilde{\eta} \right] + \eta_e \left[\frac{d}{d\tilde{s}} \int_0^1 (\tilde{\eta}^m f_\eta^2 - f_\eta) \, d\tilde{\eta} \right] + \left(\frac{df_w}{d\tilde{s}} \right) \left[m \int_0^1 \tilde{\eta}^{m-1} f_\eta \, d\tilde{\eta} - 1 \right] + m \eta_e \int_0^1 \tilde{\eta}^{m-1} f_\eta \left(\int_0^{\tilde{\eta}} \frac{\partial}{\partial \tilde{s}} f_\eta \, d\eta' \right) d\tilde{\eta} \right\} \quad (12)$$

where it has been found more convenient to introduce explicitly f_s at $\eta = \eta_e$.†

Equations (3) and (7)–(12) and the definition of $C_{0,m}$ given by equation (6) provide as many integral conditions for the velocity profile as is desired by letting $m = 0, 1, 2 \ldots.$

The energy equation and its integral conditions

Consider next the equation of energy conservation; it will be sufficient for the purposes of illustrating the method and for many problems in boundary-layer theory to assume that either a homogeneous or a gas mixture with all Lewis numbers equal to unity exists in the boundary layer. In either case the energy equation for at flow with constant stagnation enthalpy external to the boundary layer is, in terms of the Levy–Lees variables,

$$[(C/\sigma) g_\eta]_\eta + f g_\eta + 2 \tilde{m} [C (1 - \sigma^{-1}) f_\eta f_{\eta\eta}]_\eta = 2 \tilde{s} (f_\eta g_{\tilde{s}} - f_{\tilde{s}} g_\eta) \quad (13)$$

† Note that

$$f_{\tilde{s}} \Big|_{\eta=\eta_e} = \frac{\partial}{\partial \tilde{s}} \left(f_w + \int_0^{\eta} f_\eta \, d\eta \right) \Big|_{\eta=\eta_e} = \left(\frac{df_w}{d\tilde{s}} + \int_0^{\eta} \frac{\partial^2 f}{\partial \tilde{s}\, d\eta} \, d\eta \right) \Big|_{\eta=\eta_e} = \left(\frac{df_w}{d\tilde{s}} + \frac{d}{d\tilde{s}} \int_0^{\eta_e} f_\eta \, d\eta - \frac{d\eta_e}{d\tilde{s}} \right) = \left(\frac{df_w}{d\tilde{s}} + \frac{d\eta_e}{d\tilde{s}} \int_0^1 f_\eta \, d\tilde{\eta} + \eta_e \frac{d}{d\tilde{s}} \int_0^1 f_\eta \, d\tilde{\eta} - \frac{d\eta_e}{d\tilde{s}} \right).$$

Let the boundary conditions applicable at any $\tilde{s}$ be

$$\left.\begin{aligned} g(\tilde{s}, 0) &= g_w(\tilde{s}) \\ g(\tilde{s}, \infty) &= 1 \end{aligned}\right\} \quad (14)$$

Again the initial conditions at either $\tilde{s} = 0$ or at $\tilde{s} = \tilde{s}_i > 0$ will be discussed later. The Prandtl number σ as well as the transport parameter C are assumed to be given functions of the dependent variables and $\tilde{m} \equiv (u_e^2/2h_{s,e})$ to be a given function of $\tilde{s}$.

In applying a method of moments to obtain an approximate solution of equations (13) and (14) it will be assumed here that $\sigma = 0(1)$ and thus that the "edge" of the energy boundary layer corresponds roughly to the same value of η as does the edge of the momentum boundary layer. Thus a single thickness can be introduced and the method of moments involves multiplying equation (13) by $\eta^m \, d\eta$ and integrating from 0 to η_e. The first term on the left-hand side of equation (13) leads to

$$\left.\begin{aligned} &\int_0^{\eta_e} \eta^m \, [(C/\sigma)\, g_\eta]_\eta \, d\eta \\ &= -\,[\tilde{C} g_{\tilde{\eta}}]_w/\eta_e, \quad m = 0 \\ &= -\,m\, \eta_e^{m-1} \int_0^1 \tilde{\eta}^{m-1}\, \tilde{C} g_{\tilde{\eta}}\, d\tilde{\eta}, \quad m > 1 \end{aligned}\right\} \quad (15)$$

where $\tilde{C} \equiv C/\sigma$. Now the effect of the variation of C and of σ throughout the boundary layer can be taken into account as was the effect of C in the momentum equation; let

$$\tilde{C}\,(\tilde{s}, \eta) = [\tilde{C}\,(\tilde{s}, \eta) - \tilde{C}_{0,m}(\tilde{s})] + \tilde{C}_{0,m}(\tilde{s}) \quad (16)$$

so that equations (15) for $m \geqslant 1$ becomes

$$\left.\begin{aligned} &\int_0^{\eta_e} \eta^m \, [(C/\sigma)\, g_\eta]_\eta \, d\eta \\ &= -\,\tilde{C}_{01}\,(1 - g_w), \quad m = 1 \\ &= -\,m\, \eta_e^{m-1}\, \tilde{C}_{0,m}\, [1 \\ &\quad -\,(m-1) \int_0^1 \tilde{\eta}^{m-1}\, g\, d\tilde{\eta}], \quad m > 1 \end{aligned}\right\} \quad (17)$$

provided $\tilde{C}_{0,m}$ is selected so that

$$\tilde{C}_{0,1} = \frac{\int_0^1 \tilde{C} g_{\tilde{\eta}}\, d\tilde{\eta}}{(1 - g_w)}, \quad m = 1$$

and

$$\tilde{C}_{0,m} \equiv \frac{\int_0^1 \tilde{\eta}^{m-1}\, \tilde{C} g_{\tilde{\eta}}\, d\tilde{\eta}}{1 - (m-1) \int_0^1 \tilde{\eta}^{m-2}\, g\, d\tilde{\eta}}, \quad m > 1 \quad (18)$$

As above, the point of view with respect to $\tilde{C}_{0,m}$ given by equation (18) is that in the numerical integration of the ordinary differential equations, which result from the method of moments, its value is determined in general by numerical integration at the previous value of $\tilde{s}$.

The second term on the right-hand side of equation (13) leads to

$$\int_0^{\eta_e} \eta^m\, f g_\eta\, d\eta = \eta_e^m\, [f_e - \eta_e \int_0^1 \tilde{\eta}^m\, f_\eta\, g\, d\tilde{\eta} - m \int_0^1 \tilde{\eta}^{m-1}\, g f\, d\tilde{\eta}], \quad (19)$$

f and f_e being given by equations (9) and (10); the third term leads to

$$2\,\tilde{m} \int_0^{\eta_e} \eta^m\, [C\,(1 - \sigma^{-1})\, f_\eta\, f_{\eta\eta}]_\eta\, d\eta = -\,2\tilde{m}\, m\, \eta_e^{m-1} \int_0^1 \tilde{\eta}^{m-1}\, C\,(1 - \sigma^{-1})\, f_\eta\, (f_\eta)_{\tilde{\eta}}\, d\tilde{\eta} \quad (20)$$

As above define

$$\bar{C} \equiv C - \tilde{C} = [\bar{C}(\tilde{s}, \eta) - \bar{C}_{0,m}(\tilde{s})] + \bar{C}_{0,m}(\tilde{s});$$

then equation (20) yields

$$2\tilde{m} \int_0^{\eta_e} \eta^m\, [C(1 - \sigma^{-1})\, f_\eta\, f_{\eta\eta}]_\eta\, d\eta = -\,2\tilde{m}\, m\, \eta_e^{m-1}\, \bar{C}_{0,m}\, \{1 - [(m-1)/2] \int_0^1 \tilde{\eta}^{m-2} f_\eta^2\, d\tilde{\eta}\} \quad (21)$$

provided

$$\bar{C}_{0,m} = \frac{\int_0^1 \tilde{\eta}^{m-1}\, \bar{C} f_\eta^2\, d\tilde{\eta}}{2 - (m-1) \int_0^1 \tilde{\eta}^{m-2} f_\eta^2\, d\tilde{\eta}} \quad (22)$$

Finally the right-hand side of equation (13) becomes

$$2\tilde{s}\int_0^{\eta_e}\eta^m(f_\eta g_{\tilde{s}}-f_{\tilde{s}}g_\eta)\,d\eta$$
$$=2\tilde{s}\left\{\left(\frac{d\eta_e}{d\tilde{s}}\right)\left[\int_0^1 f_\eta(g-1)\,d\tilde{\eta}\right]+\eta_e\frac{d}{d\tilde{s}}\left[\int_0^1 f_\eta(g-1)\,d\tilde{\eta}\right]+\left(\frac{df_w}{d\tilde{s}}\right)(g_w-1)\right\},\quad m=0$$
$$=2\tilde{s}\,\eta_e^m\left\{\left(\frac{d\eta_e}{d\tilde{s}}\right)\left[(m+1)\int_0^1\tilde{\eta}^m f_\eta g\,d\tilde{\eta}-\int_0^1 f_\eta\,d\tilde{\eta}\right]+\left(\frac{df_w}{d\tilde{s}}\right)\left[m\int_0^1\tilde{\eta}^{m-1}g\,d\tilde{\eta}-1\right]-\eta_e\frac{d}{d\tilde{s}}\int_0^1 f_\eta(1-g\tilde{\eta}^m)\,d\tilde{\eta}+m\,\eta_e\int_0^1\tilde{\eta}^{m-1}g\left(\int_0^{\tilde{\eta}}\frac{\partial}{\partial\tilde{s}}f_\eta\,d\eta'\right)d\tilde{\eta}\right\},\quad m\geqslant 1 \tag{23}$$

Thus equations (15) for $m = 0$, and equations (17), (19), (21) and (23) with the pertinent definitions of $\tilde{C}_{0,m}$ and $\bar{C}_{0,m}$ given by equations (18) and (22) provide the set of equations representing integral conditions for the energy equation.

General comments concerning the species equations

It should be clear from the presentation above that the method of moments employed here can be applied straightforwardly to heterogeneous boundary layers involving all Lewis numbers equal to unity. Indeed except for alterations in boundary conditions and for the creation term, the equations of species conservation in terms of mass fractions can be handled much as the energy equation above. For boundary layers with multicomponent and thermal diffusion considered, the diffusional velocities are not related in a simple fashion to the gradients of concentration and temperature. However, the determination of effective, average transport coefficients as above would appear to provide a useful numerical technique for including these effects in an approximate solution of the energy and species conservation equations.

The remainder of the analysis presented here will be confined to homogeneous boundary layers so that the momentum and energy equations supplemented by an equation of state and by a description of the transport properties in terms of g and f_η provide a satisfactory description of the flow.

To exploit the integral conditions developed above for the momentum and energy equations it is necessary to assume profiles for f_η and g with a finite edge at $\eta = \eta_e$, $\tilde{\eta} = 1$; polynomials provide convenient functions therefor, although they lead to certain restrictions in application. Because of the availability of an infinite number of moment conditions a wide variety of profiles can be selected. These profiles can involve as many arbitrary functions of $\tilde{s}$ and can satisfy as many boundary conditions at $\tilde{\eta} = 0$, 1 as desired; thus a means for successive improvements of the integral method with a simple weighing function is available.

For the purposes of illustrating the method the profiles employed here will be

$$f_\eta = (2\tilde{\eta} - 2\tilde{\eta}^3 + \tilde{\eta}^4) - (a_2/3)(\tilde{\eta} - 3\tilde{\eta}^2 + 3\tilde{\eta}^3 - \tilde{\eta}^4) \tag{24}$$

$$g = 1 - (1 - g_w)(1 - 10\tilde{\eta}^3 + 15\tilde{\eta}^4 - 6\tilde{\eta}^5) + b_1(\tilde{\eta} - 6\tilde{\eta}^3 + 8\tilde{\eta}^4 - 3\tilde{\eta}^5) + b_2(\tilde{\eta}^2 - 3\tilde{\eta}^3 + 3\tilde{\eta}^4 - \tilde{\eta}^5) \tag{25}$$

where a_2, b_1 and b_2 are profile parameters which are unknown functions of $\tilde{s}$. The wall enthalpy parameter g_w is assumed to be a given function of $\tilde{s}$. Examination of the $\tilde{\eta}$-functions in these profiles indicates that the boundary conditions given by equations (2) and (14) are satisfied for all values of these parameters. Note that the velocity profile is the usual one parameter profile for flows with pressure gradient; here, there is followed the point of view suggested by Tani [14] and others, namely, that the parameter a_2 be determined not by a boundary condition at $\tilde{\eta} = 0$ but rather by an integral condition. Similarly, the parameter b_2 will not be determined here by a boundary condition but by an integral condition. It is noted that by adding one additional profile parameter to each profile the usual boundary conditions at $\tilde{\eta} = 0$ on both profiles could be satisfied without increasing the number of profile parameters which are determined by integral conditions.

With the profiles given by equations (24) and (25) the boundary-layer characteristics are known in terms of $\tilde{s}$ and η when η_e, a_2, b_1 and b_2 are obtained as functions of $\tilde{s}$. Thus, four integral conditions given by $m = 0, 1$ for both the momentum and energy equations are required.

It is convenient to consider now the quantities, which are of practical interest in the solution of a particular problem, and which may, for the specific profiles employed here, be expressed in terms of η_e and of the profile parameters; for example, the skin friction coefficient is

$$c_f = 2\tau_w/\rho_e u_e^2 = \frac{2C_w \mu_e r^j}{(2\tilde{s})^{\frac{1}{2}}\eta_e}[2 - (a_2/3)] \tag{26}$$

The heat transfer in the form of a Stanton number can be expressed as

$$\frac{q_w}{\rho_e u_e h_{s,e}(1 - g_w)} = \frac{C_w}{\sigma_w}\frac{\mu_e r^j}{(2\tilde{s})^{\frac{1}{2}}}\frac{(b_1/\eta_e)}{(1 - g_w)} \tag{27}$$

The boundary-layer characteristics which can be found explicitly by application of the inverse transformation, $\eta \to y$, are

$$\frac{\rho_e u_e r^j \delta}{(2\tilde{s})^{\frac{1}{2}}} = \eta_e \int_0^1 \frac{\rho_e}{\rho}\,d\tilde{\eta} \tag{28}$$

$$\frac{\rho_e u_e r^j \theta}{(2\tilde{s})^{\frac{1}{2}}} = \eta_e \int_0^1 f_\eta(1 - f_\eta)\,d\tilde{\eta} \tag{29}$$

$$\frac{\rho_e u_e r^j \delta^*}{(2\tilde{s})^{\frac{1}{2}}} = \eta_e \int_0^1 \left(\frac{\rho_e}{\rho} - f_\eta\right)d\tilde{\eta} \tag{30}$$

In addition to the profiles presented above it is necessary, in order to complete the system of equations, to provide the auxiliary functions related to the transport processes and to the mass density ratio, ρ_e/ρ. It will be sufficient for purposes of illustrating the method and for many boundary-layer problems to take an approximate equation of state in the form

$$\rho_e/\rho = h/h_e, \tag{31}$$

to let the Prandtl number be constant, and to let

$$C = (h/h_e)^n \tag{32}$$

where n is a constant. In terms of f_η and g the static enthalpy ratio is readily expressed as

$$\frac{h}{h_e} = \frac{g - \tilde{m}f_\eta^2}{1 - \tilde{m}} \tag{33}$$

A particular boundary-layer flow is characterized by either the distributions of or the values of β, $\tilde{m}$, f_w and g_w depending on whether the flow is respectively nonsimilar or similar. For either type of flow these flow parameters must be considered known.†

General remarks concerning the character of the final equations

With the profiles given by equations (24) and (25) the integrals which arise in the integral conditions for the momentum and energy equation can be evaluated in terms of the profile parameters and their derivatives. The details thereof are given in the appendix to this paper; it is perhaps of interest to discuss here the form and implications of the final equations.

Symbolically, the final equations arising from the integral conditions can be written in matrix form as

$$(2\tilde{s})\begin{Vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{Vmatrix}\begin{Vmatrix} \eta_e' \\ a_2' \\ b_1' \\ b_2' \end{Vmatrix} = \begin{Vmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{Vmatrix} \tag{34}$$

where $(\,)' \equiv d/d\tilde{s}$ and where the A_{ij} and B_i coefficients are functions of the dependent variables η_e, a_1, b_1, b_2 and of the flow parameters β, $\tilde{m}$, f_w and g_w and are given explicitly in the Appendix. Now for similar flows β, $\tilde{m}$, f_w and g_w must be constant and the column matrix $|B_i| = 0$. Indeed the values of η_e, a_2, b_1 and b_2

† It is implicitly assumed here that no interaction between the external and boundary-layer flow is being considered; however, the extension to the interaction case can be carried out in terms of the usual one-wave flow, which would relate β, $\tilde{m}$, $d\delta^*/d\tilde{s}$, and $d^2\delta^*/d\tilde{s}^2$.

making the column matrix zero yield the approximate solution for the similar flow related to the specified values of the flow parameters. For nonsimilar flows the column matrix will be non-zero and will provide the "forcing functions" for the dependent variables.

It is now convenient to discuss the initial conditions which can be imposed; two situations arise. If the analysis is initiated at a finite $\tilde{s} = \tilde{s}_i > 0$, then arbitrary initial values of η_e, a_2, b_1 and b_2 can be imposed. If on the contrary, integration is to begin at $\tilde{s} = 0$, it is clear from equation (34) that finite derivatives η_e', a_2', etc., can prevail only if the column matrix $|B_i| = 0$, i.e. if a similar flow corresponding to the specified values $\beta(0)$, $\tilde{m}(0)$, $f_w(0)$ and $g_w(0)$ exists. In this latter case arbitrary, initial values of η_e, a_1, etc., cannot be specified. If arbitrary distributions of the flow parameters are prescribed, then special care must be devoted to the accuracy of the integration in the neighborhood of $\tilde{s} = 0$. However, in most problems of practical interest $\beta'(0)$, $\tilde{m}'(0)$, $f_w'(0)$ and $g_w'(0)$ are zero so that $\eta_e'(0)$, $a_2'(0)$, etc., are also zero; then straightforward integration from the starting point $\tilde{s} = 0$ can be employed.

Several further points are perhaps worth noting; equation (34) can be integrated by standard techniques, as for example, by the Kutta–Runge–Gill procedure, on modest size computers. With increasing values of $\tilde{s}$ the effective transport parameters $C_{0,m}$, $\tilde{C}_{0,m}$ and $\bar{C}_{0,m}$ may be determined by numerical quadrature, e.g. by Simpson's rule, from the values of the dependent variables at the end of the previous step. Finally, the program for solving nonsimilar problems, i.e. where either any one or all of the flow parameters are varying with $\tilde{s}$ may be readily employed to find solutions to similar flows as follows: At an initial $\tilde{s} = \tilde{s}_i > 0$, assume for initial values of η_e, a_2, b_1 and b_2 estimates of the values thereof which are considered to correspond to the similar flow in question. With β, $\tilde{m}$, f_w and g_w constant integrate equation (34) for increasing $\tilde{s}$ until η_e', a_2', b_1' and b_2' become sensibly zero. The resulting values of η_e, etc., are those for the similar flow. This procedure is physically equivalent to investigating the decay to a similar flow of a boundary layer which has an initial distribution of velocity and energy and an initial thickness not corresponding to similarity.† Note that in this calculation the effect of variations in C and $\bar{C}$ through the boundary layer is taken into account in approaching the similar flow.

A detailed examination of equation (34) considered with the values for the A_{ij} coefficients given in the Appendix for the present velocity profile shows that the derivatives η_e' and a_2' become infinite when $a_2 = \pm 6$, 30, that each of these singular points correspond to simple poles, that they are independent of the variables η_e, b_1, b_2 and $\tilde{s}$, and finally that they arise from the velocity profile alone. The point $a_2 = 6$ corresponds to separation so that its occurrence corresponds to physically acceptable behavior; the point $a_2 = 30$ appears to be of no practical significance. However, there are nonsimilar flows of interest, e.g. the boundary layer on a blunt body in supersonic flow with arbitrary g_w or on a porous surface with arbitrary suction, wherein the solution satisfying the desired initial conditions leads a_2 to approach -6. Such a solution cannot be continued beyond the value of $\tilde{s}$ corresponding to $a_2 = -6$ except by artificial and unappealing means; in this case the method with the present velocity profiles must be considered to break down. For the blunt body problem this occurred with the present profiles when $g_w \geqslant 0{\cdot}3$ at various points downstream of the stagnation point; for the boundary layer on a flat plate with uniform mass transfer, $(\rho v)_w$ = constant, it is possible to integrate until $[-(\rho v)_w/\rho_e u_e \mu_e](\tilde{s}/2)^{\frac{1}{2}} \simeq 1{\cdot}5$. Thus for the analysis of these problems according to the present method, other profiles must be chosen. It is noted that the restrictions cited above do not appear relevant to similar flows and thus the present analysis provides a means for obtaining approximate solutions in such cases.

III. APPLICATIONS AND DISCUSSION OF RESULTS

In this final section the results of several similar and nonsimilar applications of the present method and profiles are described and compared to more accurate calculations where available.

† This same point of view was applied recently in [18].

Similar solution

The basic solutions for similar flows are those presented by Cohen and Reshotko [2] which apply to $C \equiv 1$. For a range of pressure gradient parameter, $\hat{\beta}$ and wall enthalpy ratio, g_w, many solutions have been obtained by the present analysis. Shown in Figs. 1 and 2 are some

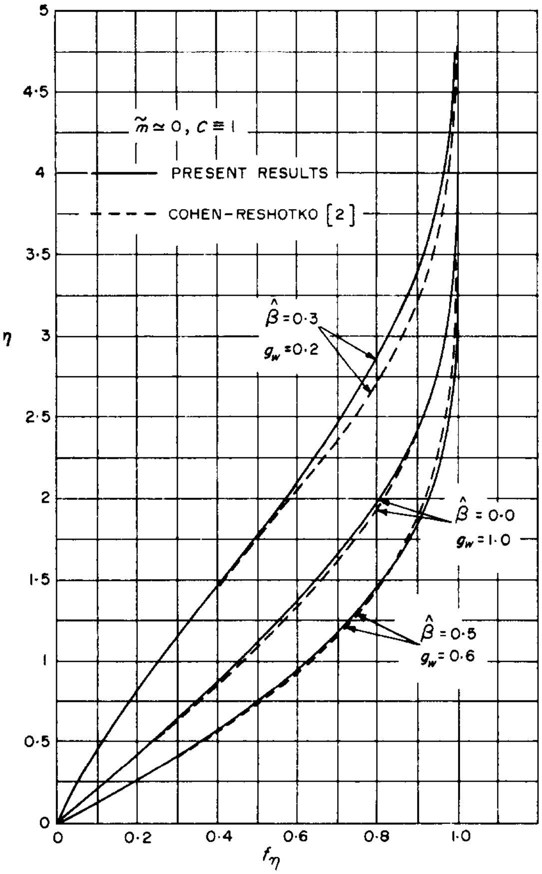

FIG. 1. Similar solutions—Velocity profiles.

typical results where comparison was made with the profiles obtained in [2]; the good agreement is noted even for the case of adverse pressure gradient. It might be remarked in passing that when the Blasius solution is used as an initial profile, approximately twenty steps in $\tilde{s}$ were required for convergence, i.e. at this point the derivatives with respect to $\tilde{s}$ were sensibly zero.

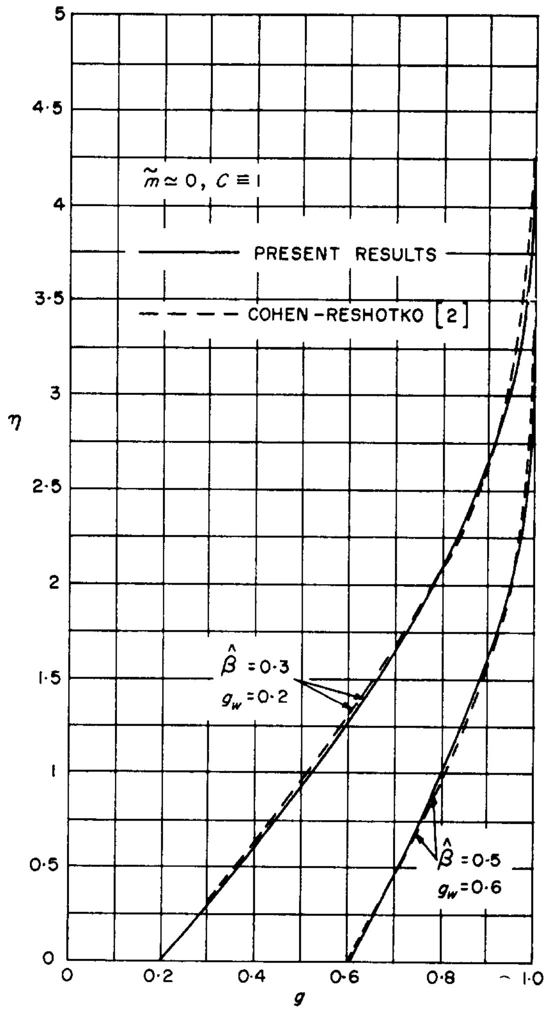

FIG. 2. Similar solutions—Enthalpy ratio.

Additional similar solutions corresponding to mass transfer with $C \equiv \sigma \equiv 1$ have been compared with the results of Emmons and Leigh [19]; the agreement indicated on Figs. 1 and 2 was again obtained.

In order to assess the treatment of variable transport properties afforded by the present analysis several calculations of air injection at an axisymmetric stagnation point ($\hat{\beta} = \frac{1}{2}$, $\tilde{m} \simeq 0$) with $C = g^{-0 \cdot 252}$, $\sigma = 0 \cdot 7$ were carried out. Comparison of a typical result with [20] wherein numerical integration of the exact equations is used is shown in Fig. 3; the satisfactory agreement will be noted.

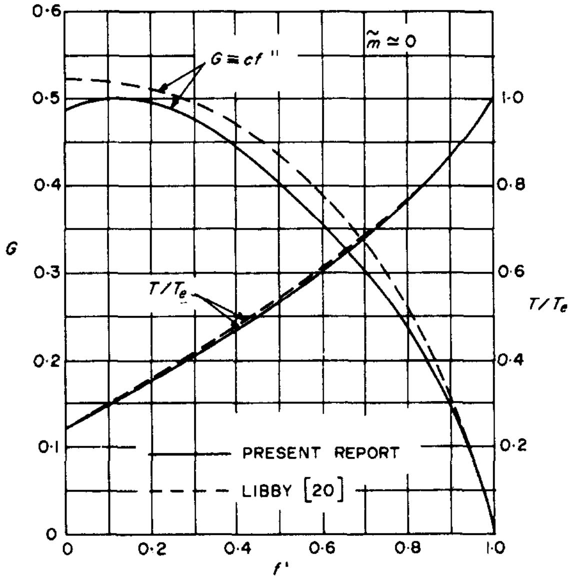

FIG. 3. Comparison of shear function and temperature ratio for mass transfer.

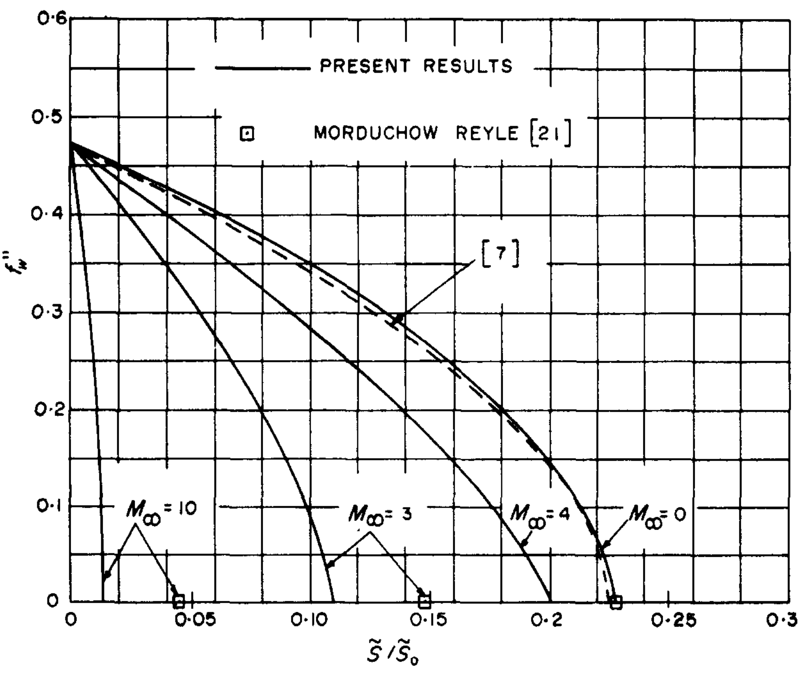

FIG. 4. Distribution of shear function for various adverse pressure gradients.

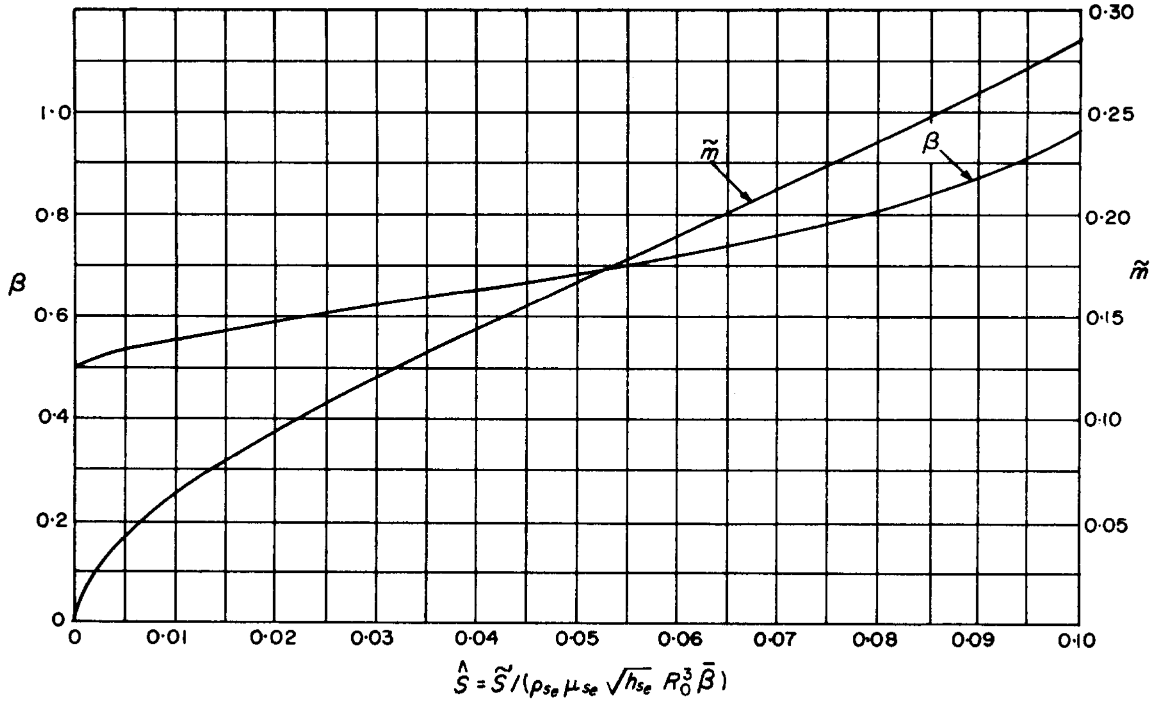

FIG. 5. Distribution of pressure gradient parameter, β, and Mach number parameter, $\tilde{m}$.

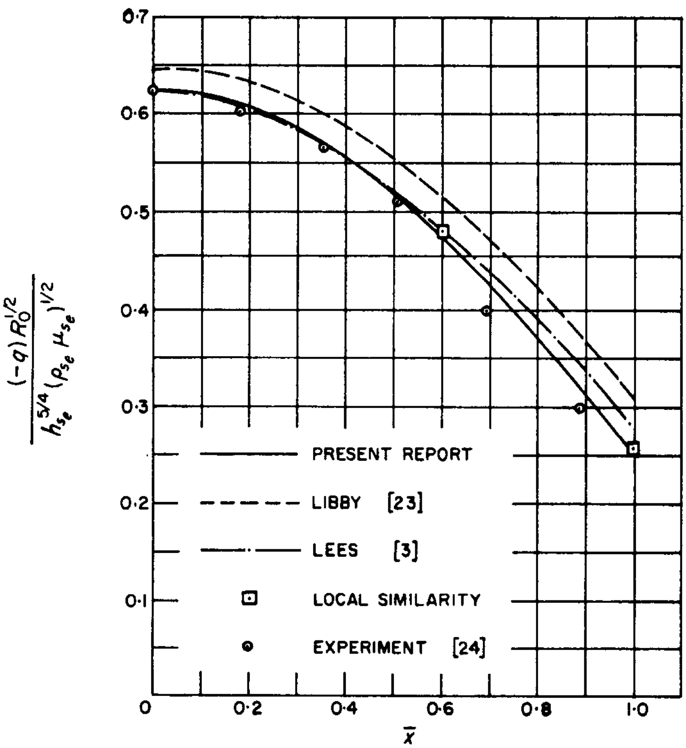

FIG. 6. Heat transfer distribution on blunt nosed body.

It can be concluded on the basis of these results that the present method provides a ready means of obtaining approximate but accurate solutions for similar flows with and without mass transfer and with and without accurately described transport properties.

Nonsimilar solutions

Several representative nonsimilar flows have been treated by the present method. The first problem considered here is a two-dimensional flow with an adverse pressure gradient such that

$$\left.\begin{aligned}(u_e/u_\infty) &= 1 - (x/x_0)\\ (x/x_0) &= 1 - [1 - (\tilde{s}/\tilde{s}_0)]^{\frac{1}{2}}\\ \tilde{m} &= \tilde{m}_\infty[1 - (\tilde{s}/\tilde{s}_0)]\\ \hat{\beta} &= (\tilde{s}/\tilde{s}_0)\{[1 - \tilde{s}/\tilde{s}_0][1 - \tilde{m}]\}^{-1}\\ g_w &= (1 - \tilde{m}_\infty)(T_w/T_\infty)\\ \tilde{m}_\infty &= [(\gamma - 1)/2]M_\infty^2\{1 + [(\gamma - 1)/2]M_\infty^2\}^{-1}\end{aligned}\right\} \quad (35)$$

where $u_\infty, x_0, \tilde{s}_0, \tilde{m}_\infty$, are reference quantities and the ratio T_w/T_∞ is specified. The purpose of this study is the prediction of the separation point for various conditions of external Mach number and wall temperature ratio.

For the case of incompressible flow,

$$M_\infty = \tilde{m}_\infty = \tilde{m} = 0,$$

Smith and Clutter [7] present the distribution of $f''_w(\tilde{s})$; a comparison with the present results is shown in Fig. 4 where the excellent agreement is noted. The results of many other investigations of the effects of $\tilde{m}_\infty$ and T_w/T_∞ are summarized in [21] and [22] in terms of the values of $(\tilde{s}/\tilde{s}_0)$ at separation. The present results for zero heat transfer and $M_\infty = 3$ and $M_\infty = 10$ are also shown in Fig. 4 along with the results given in [21]. For these latter cases it is clear that some disagreement on the separation value prevails; it should be pointed out, however, that the recent review of Morduchow [22] emphasizes the relatively wide disagreement among the various theoretical predictions of separation point for zero heat transfer and of the effect of heat transfer on separation point; in view of this situation the present disagreement should not be surprising and the present analysis should be considered to provide another theoretical prediction.

As a final calculation the case of $M_\infty = 4$, $(T_w/T_\infty) = 1{\cdot}0$ is considered; it is found that separation is predicted to occur at

$$(x/x_0) = 0{\cdot}1065.$$

This should be compared to the range of values given by other analyses from 0·173 to 0·311 (cf. [22]).

The development of the boundary layer around an axisymmetric blunt body in high speed flow is considered next. The pressure distribution was given by Newtonian from which the pressure gradient parameter $\hat{\beta}$ $(\tilde{s})$ could be computed. This resultant distribution is shown in Fig. 5. The case corresponding to a cold wall, i.e. $g_w = 0$ leads to no numerical difficulty while for $g_w \geqslant 0{\cdot}30$, the aforementioned singularity corresponding to $a_2 = -6{\cdot}0$ is encountered at a value of $\tilde{s}$ depending on g_w; the solution in these cases must be terminated and cannot be continued to larger $\tilde{s}$ values.

The heat-transfer distribution for $g_w = 0$ presented in terms of both $qR_0^{\frac{1}{2}}h_{s_e}^{-5/4}(\rho_{s_e}\mu_{s_e})^{-\frac{1}{2}}$ and (q/q_0) where q_0 is the stagnation point value is shown in Figs. 6 and 7. For comparison purposes a one-moment method [23], the Lees

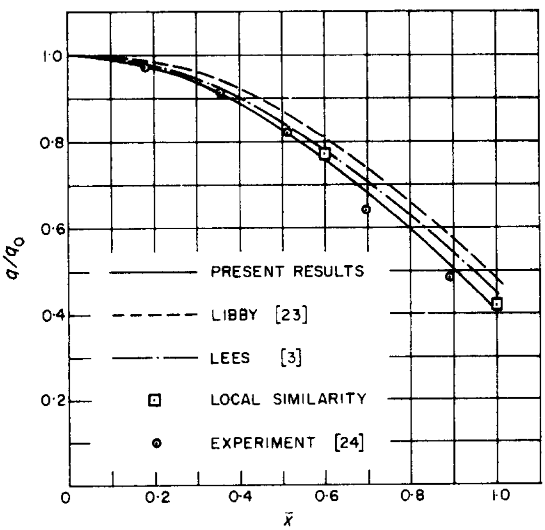

FIG. 7. Normalized heat-transfer distribution on blunt nosed body.

theory [3] and some available experimental results from [24] are also shown; in each case q_0 is of course different. The excellent agreement is noted. There are also presented some results obtained by application of local similarity. These were computed according to the present method by using the local values of β and $\tilde{m}$ and by determining the corresponding similar solution.

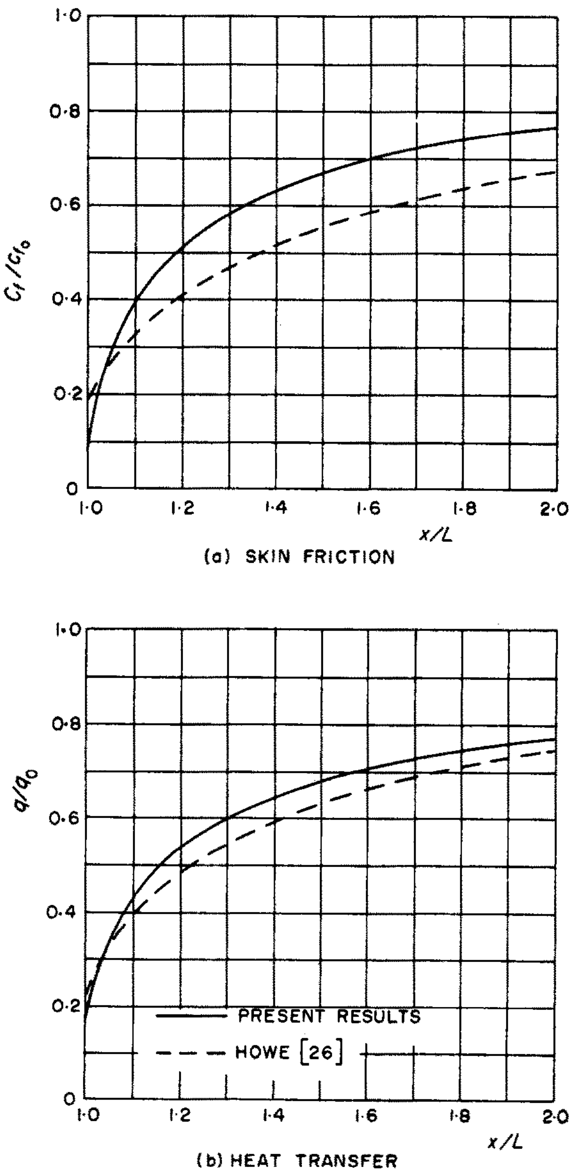

FIG. 8. Distribution of skin friction and heat transfer downstream of a permeable surface.

It is clear, as previously noted [5], that the approximations attendant with local similarity are not severe for the blunt body problem.

As a final nonsimilar problem there is considered the downstream effect of upstream transpiration cooling. A finite difference solution for the flow over a flat plate was first discussed by Howe [25] for the case of

$$f_w = -(2)^{-\frac{1}{2}}$$

in the upstream region, a constant wall temperature ratio $(T_w/T_e) = 1{\cdot}16$, and external Mach number $M_\infty = 3$. The similar solution for these conditions is obtained first and then assuming the computed values of η_e, a_2, b_1, b_2 to prevail at the start of the impermeable region the downstream solution corresponding to $f_w = 0$ is obtained.

Skin friction and heat-transfer results are shown in Fig. 8 with comparison to reference 25. The resultant c_f/c_{f_0} and q/q_0 distribution, where c_{f_0} and q_0 as computed from the similar solution corresponding to $f_w = 0$, seem to be fairly accurate.

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APPENDIX

In this appendix there will be summarized the integral conditions corresponding to $m = 0, 1$ in equations (5)–(12) and equations (15)–(23); the matrix that results therefrom [see equation (34)] when use is made of equations (31)–(33) is also presented.

The condition corresponding to $m = 0$ can be written as

$$\frac{2\tilde{s}}{\eta_e}\frac{\mathrm{d}}{\mathrm{d}\tilde{s}}(\eta_e I_1) = \frac{C_w}{\eta_e^2}[(f_\eta)_{\tilde{\eta}}]_w - I_1 - \beta I_2 - \frac{1}{\eta_e}\left[f_w + 2\tilde{s}\frac{\mathrm{d}f_w}{\mathrm{d}\tilde{s}}\right] \tag{A1}$$

$$\frac{2\tilde{s}}{\eta_e}\frac{\mathrm{d}}{\mathrm{d}\tilde{s}}(\eta_e I_7) = \frac{C_w}{\sigma_w\eta_e^2}(g_{\tilde{\eta}})_w - I_7 - \frac{1}{\eta_e}\left[f_w + 2\tilde{s}\frac{\mathrm{d}f_w}{\mathrm{d}\tilde{s}}\right](1 - g_w) \tag{A2}$$

Those corresponding to $m = 1$ are

$$\frac{2\tilde{s}}{\eta_e^2}\left[\eta_e\frac{\mathrm{d}}{\mathrm{d}\tilde{s}}(\eta_e I_6) - \frac{\mathrm{d}}{\mathrm{d}\tilde{s}}(\eta_e^2 I_4) - \eta_e^2 I_3'\right] = \frac{C_{01}}{\eta_e^2} + I_3 + I_4 - I_6 - \beta I_5 - \frac{1}{\eta_e}\left[f_w + 2\tilde{s}\frac{\mathrm{d}f_w}{\mathrm{d}\tilde{s}}\right](1 - I_6) \tag{A3}$$

$$\frac{2\tilde{s}}{\eta_e^2}\left[\eta_e\frac{\mathrm{d}}{\mathrm{d}\tilde{s}}(\eta_e I_6) - \frac{\mathrm{d}}{\mathrm{d}\tilde{s}}(\eta_e^2 I_8) - \eta_e^2 I_9'\right] = \frac{\bar{C}_{11}(1 - g_w) + \tilde{m}\,\bar{C}_{21}}{\eta_e^2} + I_9 + I_8 - I_6 - \frac{1}{\eta_e}\left[f_w + 2\tilde{s}\frac{\mathrm{d}f_w}{\mathrm{d}\tilde{s}}\right](1 - I_{10}) \tag{A4}$$

where the I_i are various integrals of $f_\eta(\tilde{\eta})$ and $g(\tilde{\eta})$ and after considerable labor can be shown to be

$$I_1 \equiv \int_0^1 f_\eta (1 - f_\eta)\, d\tilde{\eta} = (37/315) + (2\, a_2/945) - (a_2^2/2268)$$

$$I_2 \equiv \int_0^1 (g - f_\eta^2)\, d\tilde{\eta} = (263/630) + (b_1/10) + (b_2/60) - [(1 - g_w)/2] + (71 a_2^2/3780) - (a_2^2/2268)$$

$$I_3 \equiv \int_0^1 f_\eta \int_0^{\tilde{\eta}} f_\eta\, d\tilde{\eta}'\, d\tilde{\eta} = (49/200) - (7 a_2/600) + (a_2^2/7200)$$

$$I_3' \equiv \int_0^1 f_\eta \int_0^{\tilde{\eta}} \frac{\partial^2 f}{\partial \tilde{s}\, \partial \tilde{\eta}'}\, d\tilde{\eta}'\, d\tilde{\eta} = [(-103/10800) + (a_2/7200)]\, \frac{da_2}{d\tilde{s}} - \frac{1}{\eta_e} \frac{d\eta_e}{d\tilde{s}} [I_4 - I_3]$$

$$I_4 \equiv \int_0^1 \tilde{\eta}\, f_\eta^2\, d\tilde{\eta} = (124/315) - (29/3780) a_2 + (a_2^2/7560)$$

$$I_5 \equiv \int_0^1 \tilde{\eta}\, (g - f_\eta^2)\, d\tilde{\eta} = (67/630) - [(1 - g_w)/7] + (4/105)\, b_1 + (1/140) b_2 + (29 a_2/3780) - a_2^2/7560$$

$$I_6 \equiv \int_0^1 f_\eta\, d\tilde{\eta} = (7/10) - (a_2/60)$$

$$I_7 \equiv \int_0^1 f_\eta (1 - g)\, d\tilde{\eta} = (1 - g_w)\, [(17/70) - (31/2520)\, a_2] - b_1\, [(79/1260) - (19/7560)\, a_2] - b_2\, [(29/2520) - (a_2/2520)]$$

$$I_8 \equiv \int_0^1 \tilde{\eta}\, f_\eta\, g\, d\tilde{\eta} = (13/30) - (a_2/180) - (1 - g_w)\, [(332/3465) - (281/83160)\, a_2] - b_1\, [(-389/13860) + (23/27720)\, a_2] - b_2\, [(-17/3080) + (1/6930)\, a_2]$$

$$I_9 \equiv \int_0^1 g \int_0^{\tilde{\eta}} f_\eta\, d\tilde{\eta}'\, d\tilde{\eta} = (4/15) - (a_2/90) + (1 - g_w)\, [(-295/5544) + (83/23760)\, a_2] + b_1\, [(1093/69300) - (19/19800)\, a_2] + b_2\, [(289/92400) - (43/237600)\, a_2]$$

$$I_9 \equiv \int_0^1 g \int_0^{\tilde{\eta}} \frac{\partial^2 f}{\partial \tilde{s}\, \partial \tilde{\eta}'}\, d\tilde{\eta}'\, d\tilde{\eta} = [(-1/90) + (1 - g_w)\, (83/23760) - (19/19800)\, b_1 - (43/237600)\, b_2] \cdot \frac{da_2}{d\tilde{s}} - \frac{1}{\eta_e} \frac{d\eta_e}{d\tilde{s}} [I_8 - I_9]$$

$$I_{10} \equiv \int_0^1 g\, d\tilde{\eta} = 1 - [(1 - g_w)/2] + (b_1/10) + (b_2/60)$$

After some manipulation the elements occurring in the matrix [equation (34)] can be shown to be

$$A_{11} = I_1$$

$$A_{12} = \eta_e [(2/945) - (a_2/1134)]$$

$$A_{13} = 0$$

$$A_{14} = 0$$

$$B_1 = (\eta_e/2\tilde{s}) \left[\frac{C_w}{\eta_e^2} \left(2 - \frac{a_2}{3}\right) - I_1 - \beta\, I_2 - \frac{1}{\eta_e} \left(f_w + 2\tilde{s}\, \frac{df_w}{d\tilde{s}}\right) \right]$$

$$A_{21} = I_7$$

$$A_{22} = \eta_e[(-31/2520)(1 - g_w) + (19/7560)b_1 + (1/2520)b_2]$$

$$A_{23} = -\eta_e[(79/1260) - (19/7560)a_2]$$

$$A_{24} = -\eta_e[(29/2520) - (a_2/2520)]$$

$$B_2 = (\eta_e/2\tilde{s})\,[(C_w\, b_1/\sigma_w\, \eta_e^2) - I_7] + \eta_e \frac{\mathrm{d}g_w}{\mathrm{d}\tilde{s}}\,[(17/70) - (31/2520)\,a_2] - (1/2\tilde{s})\left(f_w + 2\tilde{s}\,\frac{\mathrm{d}f_w}{\mathrm{d}\tilde{s}}\right)(1 - g_w)$$

$$A_{31} = I_6 - I_4 - I_3$$

$$A_{32} = -\eta_e[(61/151200)a_2 - (41/75600)]$$

$$A_{33} = 0$$

$$A_{34} = 0$$

$$B_3 = (\eta_e/2\tilde{s})\left[(C_{01}/\eta_e^2) + I_3 + I_4 - I_6 - \bar{\beta}\, I_5 - (1/\eta_e)\left(f_w + 2\tilde{s}\,\frac{\mathrm{d}f_w}{\mathrm{d}\tilde{s}}\right)(1 - I_6)\right]$$

$$A_{41} = I_6 - I_8 - I_9$$

$$A_{42} = \eta_e[-(127/18480)(1 - g_w) + (31/17325)b_1 + (541/1663200)b_2$$

$$A_{43} = \eta_e[(-389/13860) + (23/27720)a_2]$$

$$A_{44} = \eta_e[(-17/3080) + (1/6930)a_2]$$

$$B_4 = (\eta_e/2\tilde{s})\,[\{\bar{C}_{11}(1 - g_w) + \tilde{m}\,\bar{C}_{21}\}/\eta_e^2 + I_9 + I_8 - I_6] + \eta_e \frac{\mathrm{d}g_w}{\mathrm{d}\tilde{s}}\,[(332/3465) - (281/83160)a_2] - (1/2\tilde{s})\left(f_w + 2\tilde{s}\,\frac{\mathrm{d}f_w}{\mathrm{d}\tilde{s}}\right)(1 - I_{10})$$

Résumé—Une méthode intégrale basée sur la technique des moments et spécialement utile pour les écoulements laminaires compressibles est présentée. Les équations aux dérivées partielles décrivant les phénomènes sont d'abord transformées à l'aide des variables de Lévy et de Lees et converties ensuite en conditions intégrales avec η^m comme facteur de pondération. On décrit une technique pour traiter la variation des propriétés de transport qui apparaît explicitement lorsque $m > 0$. Les équations résultantes sont appliquées pour $m = 0$ et 1 avec les profils en polynômes du 4ème et du 5ème degré employés ordinairement pour la vitesse et l'enthalpie d'arrêt. On a insisté sur la classification nette des écoulements en similitude ou non, dans le même cadre analytique. L'analyse est appliquée à une variété d'écoulements pour lesquels des résultats plus précis sont disponibles et on trouve qu'elle fournit dans de nombreux cas des résultats satisfaisants. On considère donc que la méthode actuelle améliore la méthode intégrale classique ($m = 0$) sans peine excessive.

Zusammenfassung—Es wird ein Integralverfahren angegeben, welches auf der Technik der Augenblickszustände basiert und für kompressible, laminare Strömungen besonders nützlich ist. Die zu Grunde liegenden partiellen Differentialgleichungen werden zuerst auf die Levy–Lees Variablen umgeformt und dann in eine integrierbare. Form mit η^m als einem bestimmten Faktor gebracht. Für die Handhabung der Änderungen der Transportgrössen, welche explizit für $m > 0$ auftreten, wird ein Verfahren angegeben. Die resultierenden Gleichungen werden für $m = 0, 1$ mit den im allgemeinen gebräuchlichen durch Polynome vierten oder fünften Grades beschriebenen Profilen für Geschwindigkeit und Staupunktsenthalpie angewendet. Die ausgeprägte Klassifizierung von ähnlichen und nichtähnlichen Strömungen innerhalb des gleichen analytischen Rahmens wird nachdrücklich betont. Die Analyse wird auf viele verschiedene Strömungen, für welche genauere Ergebnisse verfügbar sind, angewandt und in den meisten Fällen ergeben sich zufriedenstellende Resultate. Daher wird die vorliegende, leicht anzuwendende Methode als Verbesserung des üblichen Integralverfahrens ($m = 0$) angesehen.

Аннотация—Приводится интегральный метод моментов, применимый для ламинарного течения сжимаемой жидкости. Сначала дифференциальные уравнения в частных производных преобразуются к переменным Леви–Лиса, а затем сводятся к интегральным условиям с весом η^m. Описан метод учета изменения характеристик переноса, имеющего место при $m > 0$. Полученные уравнения применяются к случаю $m = 0$ при обычном задании профиля скорости и энтальпии торможения в виде полиномов 4–ой и 5–ой степени. Приводится классификация автомодельных и неавтомодельных течений в рамках той же теории. Этот метод применяется для анализа течений, для которых имеются более точные результаты, и он дает во многих случаях удовлетворительные результаты. Поэтому данный метод несколько улучшает обычный интегральный метод ($m = 0$) без больших усилий.